

INTERSECTING SPERNER FAMILIES AND THEIR CONVEX HULLS

Péter L. ERDŐS, Peter FRANKL and Gyula O. H. KATONA

Dedicated to Paul Erdős on his seventieth birthday

Received 26 January 1983

Let $\mathcal{F}$ be a family of subsets of a finite set of n elements. The vector $(f_0, \dots, f_n)$ is called the profile of $\mathcal{F}$ where f_i denotes the number of i -element subsets in $\mathcal{F}$. Take the set of profiles of all families $\mathcal{F}$ satisfying $F_1 \not\subset F_2$ and $F_1 \cap F_2 \neq \emptyset$ for all $F_1, F_2 \in \mathcal{F}$. It is proved that the extreme points of this set in $\mathbf{R}^{n+1}$ have at most two non-zero components.

1. Definitions, results

1.1. Convex hull of the Sperner families. Let X be a finite set of n elements and $\mathcal{F}$ be a family of its subsets ($\mathcal{F} \subset 2^X$). Then $\mathcal{F}_k$ denotes the subfamily of the k -element subsets in $\mathcal{F}$: $\mathcal{F}_k = \{A: A \in \mathcal{F}, |A|=k\}$. Its size $|\mathcal{F}_k|$ is denoted by f_k . The vector $(f_0, f_1, \dots, f_n)$ in the $(n+1)$ -dimensional Euclidean space $\mathbf{R}^{n+1}$ is called the *profile* of $\mathcal{F}$.

If α is a finite set in $\mathbf{R}^{n+1}$, the *convex hull* $\langle \alpha \rangle$ of α is the set of all convex linear combinations of the elements of α . We say that $e \in \alpha$ is an extreme point of α iff e is not a convex linear combination of elements of α different from e . It is easy to see that $\langle \alpha \rangle$ is equal to the set of all convex linear combinations of its extreme points. That is, the determination of the convex hull of a set is equivalent to finding its extreme points.

$\mathcal{F}$ is a *Sperner-family* iff it contains no members A, B with $A \subset B$ (*Sperner-property*). Consider the set σ of all profiles of the *Sperner-families*. The elements of σ can be perfectly characterized by a sequence of complicated inequalities (see [2], [3]). Sometimes it might be more useful to determine a small convex set containing σ . The best one of them is, of course, $\langle \sigma \rangle$. We find $\langle \sigma \rangle$ determining its extreme points (the extreme points of $\langle \alpha \rangle$ are briefly called extreme points of α):

Theorem 1. *The extreme points of the set σ of the profiles of the Sperner-families are*

$$(1) \quad \begin{aligned} z &= (0, 0, \dots, 0) \\ v_i &= \left(\underbrace{0, 0, \dots, 0}_{\widehat{0} \quad \widehat{1} \quad \dots} \underbrace{\binom{n}{i}}_{\widehat{i}} \underbrace{0, \dots, 0}_{\dots \quad \widehat{n}} \right) \quad (0 \leq i \leq n). \end{aligned}$$

Proof. We will show that this is nothing else but the well-known LYM-inequality ([8], [9], [12]):

$$(2) \quad \sum_{i=0}^n \frac{f_i}{\binom{n}{i}} \leq 1.$$

We have to prove two statements:

- (a) any element $(f_0, \dots, f_n)$ is a convex combination of vectors of form (1).
 (b) these latter ones are extreme points.

(a) means, by definition, that $(f_0, \dots, f_n)$ is a linear combination of z and v_i with some non-negative coefficients $\lambda, \lambda_0, \lambda_1, \dots, \lambda_n$ satisfying

$$\lambda + \sum_{i=0}^n \lambda_i = 1.$$

The choice $\lambda_i = f_i / \binom{n}{i}$ ($0 \leq i \leq n$), $\lambda = 1 - \sum_{i=0}^n f_i / \binom{n}{i}$ satisfies these conditions by (2).

Part (b) is also easy. z is an extreme point since all other elements of σ have non-negative coordinates with at least one positive one. Their convex combination cannot be z . On the other hand, if $\mathcal{F}$ is a Sperner-family then $|\mathcal{F}_i| \leq \binom{n}{i}$ holds with equality only if $\mathcal{F}$ consists of all i -element subsets. Therefore, if $u \in \sigma$ then its i -th coordinate is $\leq \binom{n}{i}$ with equality only for v_i . Hence v_i is an extreme point. ■

1.2. Intersecting Sperner-families. A family is an *intersecting* family if $A, B \in \mathcal{F}$ implies $A \cap B \neq \emptyset$. A classical theorem concerning intersecting families is the

Erdős—Ko—Rado theorem [4]. *If $\mathcal{F}$ is an intersecting family of k -element ($k \leq n/2$) subsets of an n -element set then*

$$\max |\mathcal{F}| = \binom{n-1}{k-1}. \quad \blacksquare$$

Let μ denote the set of profiles of the intersecting Sperner-families. There exist some inequalities in the literature trying to give good necessary conditions for the elements of μ . First Bollobás [1] proved

$$(3) \quad \sum_{1 \leq i \leq n/2} \frac{f_i}{\binom{n-1}{i-1}} \leq 1$$

later Greene, Katona and Kleitman [5] found

$$(4) \quad \sum_{1 \leq i \leq n/2} \frac{f_i}{\binom{n}{i-1}} + \sum_{n/2 < j \leq n} \frac{f_j}{\binom{n}{j}} \leq 1$$

for any $(f_0, f_1, \dots, f_n)$. Both inequalities are far from describing the convex hull of μ . The main aim of the present paper is to determine the convex hull or in other words the extreme points of μ .

Theorem 2. *The extreme points of the set μ of the profiles of intersecting Sperner families have at most two positive coordinates, more precisely, the extreme points are*

$$\begin{aligned} z &= (0, 0, \dots, 0), \\ v_j &= \left(\underset{\widehat{0}}{0}, \underset{\widehat{1}}{0}, \dots, \underset{\widehat{j}}{\binom{n}{j}}, \dots, \underset{\widehat{n}}{0} \right) \quad (n/2 < j \leq n), \\ w_i &= \left(\underset{\widehat{0}}{0}, \underset{\widehat{1}}{0}, \dots, \underset{\widehat{i}}{\binom{n-1}{i-1}}, \dots, \underset{\widehat{n}}{0} \right) \quad (1 \leq i \leq n/2), \\ w_{ij} &= \left(\underset{\widehat{0}}{0}, \underset{\widehat{1}}{0}, \dots, \underset{\widehat{i}}{\binom{n-1}{i-1}}, \dots, \underset{\widehat{j}}{\binom{n-1}{j}}, \dots, \underset{\widehat{n}}{0} \right) \quad (1 \leq i \leq n/2, \quad i+j > n). \end{aligned}$$

There is another way to describe the convex hull $\langle \mu \rangle$. Namely, we could list the hyperplanes bordering it. Some of them are trivial because they separate the positive orthant from the other ones, only. The next theorem presents a set of inequalities. The inequalities representing the non-trivial bordering hyperplanes are among them. Sometimes they are more applicable than the form given in Theorem 2. Anyway, we will deduce Theorem 2 from this theorem:

Theorem 3.

$$(5) \quad \sum_{1 \leq i \leq n/2} (1 - y_{n-i+1}) \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leq n-1} y_j \frac{f_j}{\binom{n-1}{j}} \leq 1$$

for any $(f_0, f_1, \dots, f_n) \in \mu$ and for any sequence $y_{\lfloor n/2 \rfloor + 1} \leq y_{\lfloor n/2 \rfloor + 2} \leq \dots \leq y_n \leq 0$ satisfying

$$(6) \quad y_j \leq 1 - \frac{j}{n} \quad (n/2 < j \leq n).$$

Observe that (5) gives (3) and (4) in the cases $y_{\lfloor n/2 \rfloor + 1} = \dots = y_n = 0$ and $y_j = 1 - j/n$ ($n/2 < j \leq n$), resp.

1.3. Weighted extremal hypergraphs. The classical theorem of Sperner [11] states that a Sperner-family on n elements cannot have more than $\binom{n}{n/2}$ members. The analogous question for intersecting Sperner-families was solved by Milner [10]. Their maximal size is $\binom{n}{\lfloor n/2 \rfloor + 1}$. Let $c(i)$ ($0 \leq i \leq n$) be a given real function. We may need to maximize $\sum_{i=0}^n c(i) |\mathcal{F}_i|$, rather than $|\mathcal{F}| = \sum_{i=0}^n |\mathcal{F}_i|$, for a certain class of

families $\mathcal{F}$. The solution of this question for Sperner-families was a folklore but it was formulated in [6]. We deduce it here from Theorem 1. (Earlier it was deduced from the equivalent (2).) Indeed, we have to maximize $\sum_{i=0}^n c(i)f_i$ for the elements of σ . The maximum is attained for at least one extreme point, hence

$$\max \sum_{i=0}^n c(i)f_i = \max \left\{ 0, \max_i c(i) \binom{n}{i} \right\}.$$

Analogously, Theorem 2 implies the next statement:

Theorem 4. *Given a real function $c(i)$ ($0 \leq i \leq n$) $\max \sum c(i)|\mathcal{F}_i|$ for intersecting Sperner-families $\mathcal{F}$ is attained for a family containing members of at most two different sizes, more precisely, for families with profiles listed in Theorem 2.*

1.4. An application of Theorem 2 for extremal problems for directed hypergraphs. Let X be a finite set of n elements. A *directed hypergraph* on X is a set of different sequences $(x_1, \dots, x_k)$ ($x_i \in X$, $x_i \neq x_j$ if $1 \leq i, j \leq k$, $i \neq j$) where k can vary from 0 (empty sequence) to n . The sequences are the *edges* of the directed hypergraph. The first possible extremal problem is the following: what is the maximum number of edges in a directed hypergraph if it does not contain two different edges $(x_1, \dots, x_k)$ and $(y_1, \dots, y_l)$ such that $(x_1, \dots, x_k)$ is a subsequence of $(y_1, \dots, y_l)$ (that is, $x_i = y_{j_i}$, $1 \leq j_1 < \dots < j_k \leq l$). We call these hypergraphs *directed Sperner-hypergraphs*.

Theorem 5. *The maximum number of edges of a directed Sperner-hypergraph on n elements is $n!$.*

Proof. If $x_1, \dots, x_n$ is any permutation of the elements of X then a directed Sperner-hypergraph contains at most one edge from the sequence $(x_1), (x_1, x_2), \dots, (x_1, x_2, \dots, x_n)$. Hence it cannot contain more than $n!$ edges. All the edges with n or $n-1$ elements, resp. give equality in the theorem. One can easily see that these constructions are the only ones. ■

If D is a sequence of different elements then $s(D)$ denotes the set of its elements. We may call $s(D)$ the *undirected version* of D . The next theorem answers a problem similar to that of Theorem 5.

Theorem 6. *The maximum number of the edges of a directed Sperner-hypergraph $\mathcal{H}$ satisfying the additional property*

$$\exists D, E \in \mathcal{H}: s(D) \cup s(E) = X$$

$$\text{is } (n-1)! + 1.$$

Proof. Fix an element $x \in X$. The hypergraph consisting of (x) and of all the sequences of length $n-2$ made from $X-x$ satisfies the conditions of the theorem and has $(n-1)(n-2)! + 1$ members. We have to prove that $|\mathcal{H}|$ cannot be more.

Let $\mathcal{M}$ denote the family of the maximal undirected versions of $\mathcal{H}$, that is, $\mathcal{M} = \{A: (A = s(D), D \in \mathcal{H}) \wedge \nexists E: (E \in \mathcal{H}, s(E) \supset A, s(E) \neq A)\}$. In the next row we use Theorem 5:

$$(7) \quad |\mathcal{H}| = \sum_{D \in \mathcal{H}} 1 = \sum_{A \in \mathcal{M}} |\{D: D \in \mathcal{H}, s(D) \subset A\}| \leq \sum_{A \in \mathcal{M}} |A|!.$$

$\mathcal{M}$ is obviously a Sperner-family and $A, B \in \mathcal{H}$ imply $A \cup B \neq X$. Let $\mathcal{M}^-$ denote the family of the complements of the members of $\mathcal{M}$. Then

$$(8) \quad \sum_{A \in \mathcal{M}} |A|! = \sum_{B \in \mathcal{M}^-} (n - |B|)! = \sum_{k=0}^n (n-k)! |\mathcal{M}^-|_k.$$

Here $\mathcal{M}^-$ is an intersecting Sperner-family. We may apply Theorem 4 with $c(i) = (n-i)!$. If we show that

$$(9) \quad \sum_{k=0}^n (n-k)! f_k \leq (n-1)! + 1$$

for any extreme point listed in Theorem 2 then (7), (8) and (9) prove the theorem. It is sufficient to prove (9) for v_j ($n/2 < j \leq n$) and w_{ij} ($1 \leq i \leq n/2$, $i+j > n$). If $(f_0, \dots, f_n) = v_j$ then we need the trivial inequality $(n-j)! \binom{n}{j} \leq (n-1)! + 1$. If $(f_0, \dots, f_n) = w_{ij}$

then the left hand side of (9) is $(n-i)! \binom{n-1}{i-1} + (n-j)! \binom{n-1}{j} = \frac{(n-1)!}{(i-1)!} + \frac{(n-1)!(n-j)}{j!} \leq \frac{(n-1)!}{(i-1)!} + \frac{(n-1)!(i-1)}{(n-i+1)!}$. If $i=1, 2$, then this quantity is $\leq (n-1)! + 1$. If $3 \leq i \leq n/2$ then $1/(i-1)! \leq 1/2$ and $(i-1)/(n-i+1)! \leq 1/2$ (the case $n \leq 4$ should be checked separately) are trivial and imply (9). ■

2. Proofs

2.1. Theorem 3 for cyclic permutations. We first prove Theorem 3. The method of cyclic permutations will be used. Let us fix a cyclic permutation of the elements of X and consider only those sets having consecutive elements in this cyclic permutation. These are called *consecutive sets*. The idea of the method is to prove the statement for a given cyclic permutation with the consecutive sets and then we prove the original statement by some counting argument listing all cyclic permutations [7]. So let us prove now the analogue of Theorem 3:

Lemma. *Let $\mathcal{G}$ be an intersecting Sperner-family of consecutive sets in a cyclic permutation of an n -element set and denote by g_i the number of i -element members of $\mathcal{G}$. The inequality*

$$(10) \quad \sum_{1 \leq i \leq n/2} (1 - y_{n-i+1}) \frac{g_i}{i} + \sum_{n/2 < j \leq n-1} y_j \frac{g_j}{n-j} \geq 1$$

holds for any sequence $y_{\lfloor n/2 \rfloor + 1} \geq \dots \geq y_n \geq 0$ satisfying

$$(11) \quad y_j \leq 1 - \frac{j}{n} \quad (\lfloor n/2 \rfloor < j \leq n).$$

Proof. Define $r = \min_{A \in \mathcal{G}} |A|$ and $s = n - \max_{A \in \mathcal{G}} |A|$. First we prove the lemma for $r-s \leq 1$ (Part 1) then we prove it by induction on $r-s > 1$ (Part 2).

We will suppose in the future that

$$(12) \quad r \leq n/2.$$

The opposite case $r > n/2$ is easy. Indeed, the Sperner-property implies that at most one member of $\mathcal{G}$ can start from one point of X . Therefore $|\mathcal{G}| = \sum_{\lfloor n/2 \rfloor < j \leq n-1} g_j \leq n$ holds and hence (10) follows:

$$\sum_{\lfloor n/2 \rfloor < j \leq n-1} y_j \frac{g_j}{n-j} \leq \sum_{\lfloor n/2 \rfloor < j \leq n-1} \left(1 - \frac{j}{n}\right) \frac{g_j}{n-j} = \frac{1}{n} \sum_{\lfloor n/2 \rfloor < j \leq n-1} g_j \leq 1.$$

Part I. $r - s \leq 1$. Let A_1 realize the size r , that is, $A_1 \in \mathcal{G}$, $|A_1| = r$. Denote the elements of A_1 by $\alpha_1, \alpha_2, \dots, \alpha_r$ in the order of the fixed cyclic permutation. Since $\mathcal{G}$ is a Sperner-family it can contain at most two sets with α_i as an endpoint or starting point (along the permutation) (Fig. 1). Let us denote them by E_i and S_i , resp. $\mathcal{G}$ is intersecting therefore if both E_i and S_{i+1} are defined then they must intersect "at their other end" (Fig. 2).

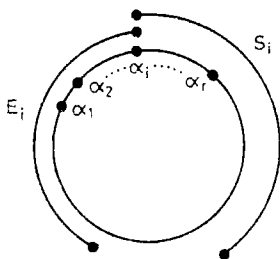


Fig. 1

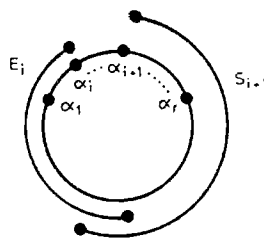


Fig. 2

This implies

$$(13) \quad |E_i| + |S_{i+1}| > n.$$

Introduce the notation

$$w(j) = \begin{cases} \frac{1 - y_{n-j+1}}{j} & \text{if } 1 \leq j \leq n/2 \\ \frac{y_j}{n-j} & \text{if } n/2 < j \leq n-1. \end{cases}$$

We shall prove the inequality

$$(14) \quad w(|E_i|) + w(|S_{i+1}|) \leq \frac{1}{r}$$

in several cases where $w(|E_i|)$ and $w(|S_{i+1}|)$ are considered to be 0 if E_i and S_{i+1} are not defined, resp.:

- (14) is trivial if none of E_i and S_{i+1} is defined.
- If one of them is defined, only (say E_i), and it has a size $\leq n/2$ then

$$w(|E_i|) = \frac{1 - y_{n-|E_i|+1}}{|E_i|} \leq \frac{1}{r}$$

follows from $|E_i| \geq r$ and $y_{n-|E_i|+1} \geq 0$.

c) If one of them (say E_i) is defined, only, and it has a size $>n/2$ then

$$w(|E_i|) = \frac{y_{|E_i|}}{n - |E_i|} \leq \frac{1}{n} \leq \frac{1}{r}$$

follows by $y_{|E_i|} \leq \frac{n - |E_i|}{n}$ (see (11)).

d) If both of them are defined and their sizes are $>n/2$ then $w(|E_i|)$, $w(|S_{i+1}|) \leq 1/n$ follow like before. Hence (14) is an easy consequence of (12).

e) Suppose now that both E_i and S_{i+1} are defined and one of them (say E_i) has a size $\leq n/2$. It follows by (13) that $|S_{i+1}| > n/2$. Then we can prove the weaker inequality

$$(15) \quad w(|E_i|) + w(|S_{i+1}|) \leq \frac{1}{r} + \frac{y_{n-r+1}}{r(r-1)}$$

instead of (14).

$$(16) \quad w(|E_i|) = \frac{1 - y_{n-|E_i|+1}}{|E_i|} \leq \frac{1 - y_{n-|E_i|+1}}{r}$$

is a consequence of the definition of r . (13) and the monotonicity of y 's imply

$$(17) \quad y_{|S_{i+1}|} \leq y_{n-|E_i|+1}.$$

By the definition of s we have $n - |S_{i+1}| \leq s \leq r - 1$. Hence and from (17) we obtain

$$w(|S_{i+1}|) = \frac{y_{|S_{i+1}|}}{n - |S_{i+1}|} \leq \begin{cases} \frac{y_{n-|E_i|+1}}{r} & \text{if } n - |S_{i+1}| \leq r \\ \frac{y_{n-r+1}}{r-1} & \text{if } n - |S_{i+1}| = r - 1. \end{cases}$$

The sum of (16) and this inequality gives (14) in the first case while in the second case we use $y_{n-|E_i|+1} \leq y_{n-r+1}$ before the summation:

$$w(|E_i|) + w(|S_{i+1}|) \leq \frac{1 - y_{n-r+1}}{r} + \frac{y_{n-r+1}}{r-1} = \frac{1}{r} + \frac{y_{n-r+1}}{r(r-1)}.$$

(15) is proved.

As any member of $\mathcal{G}$ meets A_1 and no other member can contain it, the possible members of $\mathcal{G}$ are $A_1, E_1, S_2, E_2, S_3, \dots, E_{r-1}, S_r$ (some of them might be undefined). Hence, applying (15) we obtain the inequality

$$\sum_{A \in \mathcal{G}} w(|A|) \leq w(|A_1|) + (r-1) \frac{1}{r} + \frac{y_{n-r+1}}{r} = \frac{1 - y_{n-r+1}}{r} + \frac{r-1}{r} + \frac{y_{n-r+1}}{r} = 1$$

what is nothing else but the desired (10). We have proved the lemma for $r-s \leq 1$.

Part 2. Suppose now that $t=r-s>1$ and that the lemma is proved for smaller values of $r-s$. A subfamily $A_1, \dots, A_b$ of $\mathcal{G}$ is called a *block* if $|A_1| = \dots = |A_b| = n-s$ and there are consecutive elements $\alpha_0, \alpha_1, \dots, \alpha_{b+n-s}$ (in this order along the given cyclic permutation) such that

$$A_i = \{\alpha_i, \dots, \alpha_{i+n-s-1}\} \in \mathcal{G} \quad (1 \leq i \leq b)$$

but

$$\{\alpha_0, \alpha_1, \dots, \alpha_{n-s-1}\} \notin \mathcal{G}, \quad \{\alpha_{b+1}, \dots, \alpha_{b+n-s}\} \notin \mathcal{G}.$$

We have to distinguish two cases:

2a. $b \leq s$ for any block in $\mathcal{G}$.

Define the family $\mathcal{G}^* = \{B: (|B|=n-s-1) \wedge (B \text{ consecutive}) \wedge (\exists A: A \in \mathcal{G}, |A|=n-s, A \supset B)\}$. As $\mathcal{G}$ is a Sperner-family $\mathcal{G} \cap \mathcal{G}^* = \emptyset$ follows. Let $\mathcal{G}' = (\mathcal{G} - \mathcal{G}_{n-s}) \cup \mathcal{G}^*$. It is easy to see that $\mathcal{G}'$ is a Sperner-family. On the other hand it is intersecting: $A \cap B \neq \emptyset$ ($A, B \in \mathcal{G}'$) is non-trivial only when one of them (say A) is an element of $\mathcal{G}^*$. Then $|A|=n-s-1$, $|B| \geq r$ and $r-s > 1$ imply $|A| + |B| > n$, that is, $A \cap B \neq \emptyset$.

We will need the inequality

$$(18) \quad |\mathcal{G}_{n-s}|(s+1) \leq |\mathcal{G}^*|s.$$

Let $\mathcal{G}_{n-s}$ be divided into blocks of lengths $b_1, \dots, b_u$ where

$$(19) \quad \sum_{j=1}^u b_j = |\mathcal{G}_{n-s}| \leq us$$

by the suppositions of this case. The block of length b_j induces b_j+1 members into $\mathcal{G}^*$. No element of $\mathcal{G}^*$ comes from two different blocks. Thus $|\mathcal{G}^*| = \sum_{j=1}^u (b_j+1)$.

(19) implies $(s+1) \left(\sum_{j=1}^u b_j \right) \leq s \sum_{j=1}^u (b_j+1)$ what is nothing else but (18).

The inequality

$$(20) \quad y_{n-s} \frac{g_{n-s}}{s} \leq y_{n-s-1} \frac{|\mathcal{G}^*|}{s+1}$$

follows by (18) and $y_{n-s} \leq y_{n-s+1}$. (Observe that $r-s > 1$ implies $n-s-1 > n/2$ unless $r = (n/2)+1$ which is excluded by (12).) We needed (20) for proving that the left hand side of (10) is not less for $\mathcal{G}'$ than for $\mathcal{G}$:

$$(21) \quad \sum_{r \leq i \leq n/2} (1-y_{n-i+1}) \frac{g_i}{i} + \sum_{n/2 < j \leq n-s} y_j \frac{g_j}{n-j} \\ \leq \sum_{r \leq i \leq n/2} (1-y_{n-i+1}) \frac{g_i}{i} + \sum_{n/2 < j < n-s} y_j \frac{g_j}{n-j} + y_{n-s-1} \frac{|\mathcal{G}^*|}{s+1}.$$

The largest sets in $\mathcal{G}'$ have sizes $n-s-1$, thus $s'=s+1$, $t'=r-(s+1) < t$. We may apply the induction hypothesis: (10) holds for $\mathcal{G}'$. Consequently, it also holds for $\mathcal{G}$ by (21). Case 2a is settled.

2b. $\mathcal{G}$ contains a block with $b > s$.

Choose an $A_1 \in \mathcal{G}$ with $|A_1|=r$. Let the elements of X be $\alpha_1, \alpha_2, \dots, \alpha_n$ following the cyclic permutation and suppose that $A_1 = \{\alpha_1, \dots, \alpha_r\}$. We can list all $(n-s)$ -element consecutive sets meeting but not containing A_1 :

$$\{\alpha_2, \dots, \alpha_{n-s+1}\}, \{\alpha_3, \dots, \alpha_{n-s+2}\}, \dots, \{\alpha_s, \dots, \alpha_{n-1}\}, \\ \{\alpha_{s+1}, \dots, \alpha_n\}, \{\alpha_{s+2}, \dots, \alpha_n, \alpha_1\}, \dots, \{\alpha_{r+1}, \dots, \alpha_n, \alpha_1, \dots, \alpha_{r-s}\}, \\ \{\alpha_{r+2}, \dots, \alpha_n, \alpha_1, \dots, \alpha_{r-s+1}\}, \dots, \{\alpha_{r+s}, \dots, \alpha_n, \alpha_1, \dots, \alpha_{r-1}\}.$$

In each of the first and third rows there are $s-1$ sets. On the other hand, the union of A_1 and any set in the middle row is X . It follows from the supposition of this case that some $s+1$ consecutive sets of the above sequence belong to $\mathcal{G}$. One of them belongs to the middle row. Call this set $A_2 = \{\alpha_u, \dots, \alpha_n, \alpha_1, \dots, \alpha_{u-s-1}\}$. Summarizing:

$$(22) \quad |A_1| = r, \quad |A_2| = n-s$$

$$(23) \quad A_1 \cup A_2 = X$$

$$(24) \quad \text{any point of } X-A_2 \text{ is either a starting point or an endpoint of a set } A \in \mathcal{G}, \\ |A| = n-s.$$

It is easy to check that we can have one more assumption:

$$(25) \quad A_1 \cap A_2 \text{ is a union of two non-empty intervals } I = \{\alpha_1, \dots, \alpha_{u-s-1}\} \text{ and } J = \{\alpha_u, \dots, \alpha_r\}.$$

We shall prove the following statement:

$$(26) \quad \text{there are at most } r-s+1 \text{ members of } \mathcal{G} \text{ containing } X-A_2.$$

Let $A \neq A_1$ be a member of $\mathcal{G}$ satisfying $A \supset X-A_2$. One of the endpoints of A must be in $I \cup J \cup \{\alpha_{u-1}, \alpha_{u-s}\}$ otherwise one of the conditions $A \supset X-A_2$, $A \not\subset A_2$, $A \not\subset A_1$ would be violated. Moreover, if both endpoints of A are in $I \cup J \cup \{\alpha_{u-1}, \alpha_{u-s}\}$ then they are both either in $I \cup \{\alpha_{u-s}\}$ or in $J \cup \{\alpha_{u-1}\}$. Let $e(A)$ denote the endpoint of A being in $I \cup J \cup \{\alpha_{u-1}, \alpha_{u-s}\}$ if there is only one. If there are two such endpoints let $e(A)$ denote the one being "closer" to $X-A_2$, that is, the endpoint with larger index in $I \cup \{\alpha_{u-s}\}$ and with smaller index in $J \cup \{\alpha_{u-1}\}$. It is easy to check that $e(A)$ is an injection and that $e(A)$ cannot be α_1 or α_r . Therefore $e(A)$ can have at most $|I \cup J| = |A_1 \cap A_2| = r-s$ different values. Consequently, the number of sets $A \neq A_1$, $A \supset X-A_2$, $A \in \mathcal{G}$ is at most $r-s$. Including A we obtain the bound (26).

Let us show now that

$$(27) \quad A \in \mathcal{G} \text{ implies } w(|A|) \leq (1 - y_{n-r+1})/r.$$

If $|A| \leq n/2$ then it is sufficient to substitute $|A| \leq r$ and $y_{n-|A|+1} \leq y_{n-r+1}$ into the definition of $w(|A|)$. If $|A| > n/2$ then $y_{|A|} \leq 1 - \frac{|A|}{n}$, $r \leq n/2$ and $y_{n-r+1} \leq 1 - \frac{n-r+1}{n}$ lead to

$$w(|A|) = \frac{y_{|A|}}{n-|A|} \leq \frac{1}{n} \leq \frac{1}{r} \frac{n-r+1}{n} \leq \frac{1}{r} (1 - y_{n-r+1}).$$

(27) is proved.

If $A \in \mathcal{G}$ but $A \cup A_2 \neq X$, $A \neq A_2$ then one of the endpoints of A must be in $X-A_2$ (otherwise either $A \cup A_2 = X$ or $A \subset A_2$ would follow). Since no member of $\mathcal{G}$ contains another one, any point of $X-A_2$ is an endpoint (starting point) of at most one member of $\mathcal{G}$. Altogether there are $2(s-1)$ such sets $A \in \mathcal{G}$, $A \cup A_2 \neq X$, $A \neq A_2$. $s-1$ of them are of size $n-s$ by (24). For the rest we can use (27):

$$(28) \quad \sum_{\substack{A \in \mathcal{G} \\ A \cup A_2 \neq X \\ A \neq A_2}} w(|A|) \leq (s-1) \frac{1 - y_{n-r+1}}{r} + (s-1) \frac{y_{n-s}}{s}.$$

Hence we obtain the next upper bound for the left hand side of (10):

$$\begin{aligned}\sum_{A \in \mathcal{C}} w(|A|) &= \sum_{\substack{A \in \mathcal{C} \\ A \cup A_2 = X}} w(|A|) + w(|A_2|) + \sum_{\substack{A \in \mathcal{C} \\ A \cup A_2 \neq X \\ A \neq A_2}} w(|A|) \\ &\leq (r-s+1) \frac{1-y_{n-r+1}}{r} + \frac{y_{n-s}}{s} + (s-1) \frac{1-y_{n-r+1}}{r} + (s-1) \frac{y_{n-s}}{s} \\ &= 1 - y_{n-r+1} + y_{n-s}\end{aligned}$$

where (26), (27) and (28) are used. $r-s > 1$ implies $n-r+1 < n-s$ and therefore $y_{n-r+1} > y_{n-s}$. Indeed, we obtained

$$\sum_{A \in \mathcal{C}} w(|A|) \leq 1 - y_{n-r+1} + y_{n-s} \leq 1. \quad \blacksquare$$

2.2. Proof of Theorem 3 using the cyclic permutations. Let $\mathcal{F}$ be a family with profile $(0, f_1, f_2, \dots, f_{n-1}, 0)$. The following function will be defined for any cyclic permutation $\mathcal{C}$ of X and for any $A \subset X$:

$$w(\mathcal{C}, A) = \begin{cases} w(|A|) & \text{if } A \in \mathcal{F} \text{ and } A \text{ is consecutive in } \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

We will evaluate the sum $\sum_{\mathcal{C}, A} w(\mathcal{C}, A)$ in two different ways: first fixing A , running $\mathcal{C}$ and then in the opposite order.

$$(29) \quad \sum_{\mathcal{C}, A} w(\mathcal{C}, A) = \sum_{A \in \mathcal{F}} w(|A|) |A|! (n-|A|)!$$

follows from the fact that there are $|A|!(n-|A|)!$ cyclic permutations in which A is consecutive. On the other hand

$$\sum_{\mathcal{C}, A} w(\mathcal{C}, A) = \sum_{\mathcal{C}} \sum_{\substack{A: A \in \mathcal{F} \\ A \text{ cons. in } \mathcal{C}}} w(|A|)$$

can be written. Here the last sum is ≤ 1 by the Lemma. Consequently

$$(30) \quad \sum_{\mathcal{C}, A} w(\mathcal{C}, A) \leq (n-1)!.$$

Comparing the right hand sides of (29) and (30)

$$\sum_{A \in \mathcal{F}} \frac{w(|A|)}{\frac{(n-1)!}{|A|!(n-|A|)!}} \leq 1$$

can be obtained. Substituting the definition of $w(|A|)$ this inequality gives an equivalent form of (5). $\blacksquare$

2.3. Proof of Theorem 2 using the duality theorem of linear programming. 1. First we prove that if $(f_0, f_1, \dots, f_n) \in \mu$ then there is a convex combination $(g_0, g_1, \dots, g_n)$ of z, v_j ($n/2 < j \leq n$) and w_{ij} ($1 \leq i \leq n/2, i+j > n$) satisfying $g_j \geq f_j$ ($0 \leq j \leq n$).

Let $u_{\lfloor n/2 \rfloor + 1}, \dots, u_{n-1}, u_n$ be a sequence of non-negative reals such that

$$(31) \quad u_j \leq 1 - \frac{j}{n} \quad (n/2 < j \leq n).$$

Then the sequence

$$y_j = \max_{k \geq j} u_k \quad (n/2 < j \leq n)$$

will be monotonic and preserves property (31) (e.g. (6)). On the other hand $u_j \leq y_j$ ($n/2 < j \leq n$) holds, consequently (5) is true for these y and it implies

$$(32) \quad \sum_{1 \leq i \leq n/2} \left(1 - \max_{n-i+1 \leq j \leq n} u_j\right) \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leq n-1} u_j \frac{f_j}{\binom{n-1}{j}} \leq 1.$$

Suppose that $u_i \leq 1 - \max_{n-i+1 \leq j \leq n} u_j$ ($1 \leq i \leq n/2$) or equivalently

$$(33) \quad u_i + u_j \leq 1 \quad (\text{for all } 1 \leq i \leq n/2, n-i+1 \leq j \leq n).$$

Then we can substitute u_i in the place of $1 - \max u_j$ in (32). We conclude that

$$\sum_{1 \leq i \leq n/2} u_i \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leq n-1} u_j \frac{f_j}{\binom{n-1}{j}} \leq 1$$

holds under conditions (31) and (33). The above statement can be formulated in terms of linear programming:

$$\max \left(\sum_{1 \leq i \leq n/2} u_i \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leq n-1} u_j \frac{f_j}{\binom{n-1}{j}} \right) \leq 1$$

under constraints (33) and

$$(34) \quad u_j \frac{n}{n-j} \leq 1 \quad (n/2 < j \leq n-1) \quad u_n \leq 0, \quad u_j \geq 0 \quad (n/2 \leq j \leq n).$$

Consider the dual problem. We associate the variables μ_j with constraints (34) and v_{ij} with (33):

$$(35) \quad \min \left(\sum_{n/2 < j \leq n-1} \mu_j + \sum_{1 \leq i \leq n/2} \sum_{n-i+1 \leq j \leq n} v_{ij} \right) \leq 1$$

under the constraints

$$(36) \quad \sum_{n/2 < j \leq n-1} v_{ij} \leq \frac{f_i}{\binom{n-1}{i-1}} \quad (1 \leq i \leq n/2),$$

$$(37) \quad \sum_{1 \leq i \leq n/2} v_{ij} + \mu_j \frac{n}{n-j} \leq \frac{f_j}{\binom{n-1}{j}} \quad (n/2 < j \leq n-1),$$

$$(38) \quad \sum_{1 \leq i \leq n/2} v_{in} + \mu_n \leq 0 \quad \text{and}$$

$$v_{ij} \geq 0, \quad \mu_j \geq 0 \quad (1 \leq i \leq n/2, \quad n/2 < j \leq n).$$

(38) is superfluous, (36) and (37) can be rewritten in the forms

$$(39) \quad \sum_{n-i+1 \leq j \leq n-1} v_{ij} \binom{n-1}{i-1} \leq f_i \quad (1 \leq i \leq n/2, \quad n-i+1 \leq j \leq n)$$

and

$$(40) \quad \sum_{1 \leq i \leq n/2} v_{ij} \binom{n-1}{j} + \mu_j \binom{n}{j} \leq f_j \quad (n/2 < j \leq n-1).$$

Let us concise (39) and (40) into a vectorial form:

$$(41) \quad \sum_{1 \leq i \leq n/2} v_{ij} w_{ij} + \sum_{n/2 < j \leq n-1} \mu_j v_j \leq (f_1, f_2, \dots, f_{n-1})$$

(where w_{ij} and v_j are truncated; their first and last coordinates are omitted). We obtained that under constraint (41) (35) has a solution ≤ 1 . In other words, there are non-negative v 's and μ 's satisfying (41) with a sum ≤ 1 . (41) can be easily completed with the 0th and n th coordinates: 1) $f_0=0$ since $\emptyset$ cannot be a member of an intersecting family ($\emptyset \cap \emptyset = \emptyset$); 2) if $f_n=0$ then the situation is the same; if $f_n=1$ then $f_0=\dots=f_{n-1}=0$ by the Sperner-property and hence $\mu_n=1$ is suitable.

Multiplying all v_{ij} and μ_j with the appropriate constant (≥ 1) their sum will be equal to 1 as desired.

2. In the first part of the proof we proved that there is a convex combination $(g_0, \dots, g_n)$ of the vectors v_j and w_{ij} for any given $(f_0, \dots, f_n)$ such that

$$(42) \quad g_i \leq f_i \quad (0 \leq i \leq n).$$

Choose $(g_0, \dots, g_n)$ maximizing the number of coordinates with equality in (42). Suppose that this number is $< n+1$ and $g_t > f_t$. The vector $(g_0, \dots, g_{t-1}, 0, g_{t+1}, \dots, g_n)$ is also a convex combination of the vectors $z, v_j, w_i, w_{i,j}$; we have to change the t th coordinate of each vector for 0; the set $z, v_j, w_i, w_{i,j}$ is closed under this operation. (This is the first place where the vectors z and w_i are used.) $(g_0, \dots, g_{t-1}, f_t, g_{t+1}, \dots, g_n)$ is a convex combination of $(g_0, \dots, g_{t-1}, 0, g_{t+1}, \dots, g_n)$ and $(g_0, \dots, g_n)$ (since $0 \leq f_t \leq g_t$), therefore it is a convex combination of z, v_j, w_i

and w_{ij} . This new vector $(g_0, \dots, g_{i-1}, f_i, g_{i+1}, \dots, g_n)$ has more common coordinates with $(f_0, \dots, f_n)$ than $(g_0, \dots, g_n)$ does. This contradiction leads to the statement that $(f_0, \dots, f_n)$ itself is a convex combination of the vectors z, v_j, w_i and w_{ij} .

3. In the second part of the proof we proved that only the vectors listed in Theorem 2 can be extreme points of μ . Now we have to verify that they are really extreme points. This is trivial for z .

It is easy to construct an intersecting Sperner-family with profile w_i ($1 \leq i \leq n/2$): take all the i -element subsets containing a fixed element of the ground set. On the other hand, the Erdős—Ko—Rado theorem implies that if $(f_0, \dots, f_n) \in \mu$ then $f_i \leq \binom{n-1}{i-1}$. Hence if w_i is a convex combination of some vectors from μ then they all must have $f_i = \binom{n-1}{i-1}$. Similarly, their other coordinates are necessarily 0. The only such vector is w_i . One can see in the same way that v_j ($n/2 < j \leq n$) is in μ and it is an extreme point of μ .

The construction of an intersecting Sperner-family with profile w_{ij} : take all i -element subsets containing a fixed element x and all j -element subsets not containing x . Suppose that w_{ij} is a convex combination of some elements of μ . As above, all of them must have $\binom{n-1}{i-1}$ in the i th coordinate. The only intersecting Sperner-family with $\binom{n-1}{i-1}$ i -element sets is the above construction of all i -element subsets containing x . No j -element set can contain x . It would then contain an i -element set as a subset. Hence $(f_0, \dots, f_n) \in \mu$ and $f_i = \binom{n-1}{i-1}$ imply $f_j \leq \binom{n-1}{j}$. Therefore all the vectors in the convex combination must have $\binom{n-1}{j}$ as j th coordinate. Like above, the other coordinates are 0. w_{ij} is the only such vector, therefore it is really an extreme point of μ .

References

- [1] B. BOLLOBÁS, Sperner systems consisting of pairs of complementary subsets, *J. Combinatorial Theory A* **15** (1973), 363—366.
- [2] G. F. CLEMENTS, A minimization problem concerning subsets of a finite set, *Discrete Math.* **4** (1973), 123—128.
- [3] D. E. DAYKIN, J. GODFREY and A. J. W. HILTON, Existence theorems for Sperner families, *J. Combinatorial Theory A* **17** (1974), 245—251.
- [4] P. ERDŐS, CHAO KO and R. RADO, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford (2)* **12** (1961), 313—318.
- [5] C. GREENE, G. O. H. KATONA and D. J. KLEITMAN, Extensions of the Erdős—Ko—Rado theorem, *SIAM* **55** (1976), 1—8.
- [6] G. O. H. KATONA, Two applications of Sperner type theorems, *Periodica Math. Hungar.* **3** (1973), (3) 19—26.
- [7] G. O. H. KATONA, A simple proof of Erdős—Ko—Rado theorem, *J. Combinatorial Theory B* **13** (1972), 183—184.
- [8] D. LUBELL, A short proof of Sperner's Lemma, *J. Combinatorial Theory* **1** (1966), 299.
- [9] L. D. MESHALKIN, A generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Veroyatnost. i Primen.* **8** (1963), 219—220 (in Russian).

- [10] E. C. MILNER, A combinatorial theorem on systems of sets, *J. London Math. Soc.* **43** (1968), 204—206.
- [11] E. SPERNER, Ein Satz über Untermenge einer endlichen Menge, *Math. Z.* **27** (1928), 544—548.
- [12] K. YAMAMOTO, Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343—353.

Péter L. Erdős

*Institute of Mathematics and
Computer Science
K. Marx University of Economics
Budapest, Pf. 482, H-1828, Hungary*

Peter Frankl

*CNRS
15 Quai Anatole France
75007 Paris, France*

Gyula O. H. Katona

*Mathematical Institute of the
Hungarian Academy of Sciences
Budapest, Pf. 428, H-1395, Hungary*